Quantum dynamics with disorder:
Localization, and how it is (not) related to glassiness.
I. MBL is not a quantum Glass.

- Quantum Disordered Systems: Frameworks
- MBL: A tentative definition
  - The Ingredients
  - The Questions
  - The Picture
- MBL is not a Glass with $\epsilon > 0$.

II. Localization is glassy, though.

- Computing Decay Rates:
- The directed polymer problem
- Localization is glassy, though
EQUILIBRIUM: $\mathcal{Z}_B = \text{tr}\{\tilde{\rho} e^{-\beta \hat{H}}\}$

Quantum Disordered Systems: Frameworks

OPEN SYSTEMS

ISOLATED SYSTEMS
Quantum Disordered Systems: Frameworks

EQUILIBRIUM: \( Z_B = \text{tr} \{ e^{-\beta H} \} \)
- quantum disordered system + bath
  - (Quantum) phase transitions
  - Langevin dynamics
- e.g. Bose glass [GIAMARCHI, SCHULZ], [TALK THIBAUD M.]

OPEN SYSTEMS

ISOLATED SYSTEMS
Quantum Disordered Systems: Frameworks

EQUILIBRIUM: $\Xi_B = \text{tr} \xi e^{-\beta H}$
- quantum disordered system + bath
  - (Quantum) phase transitions
  - Langevin dynamics
- e.g. Bose glass [Giamarchi, Schulz], [Talk Thibaud M.]

OPEN SYSTEMS
- quantum disordered system + driving + bath
  - stationary states
  - interplay bath-driving
- e.g. [Maimbourg, Rosso et al.]
  [Talk Saptarshi M.]
Quantum Disordered Systems: Frameworks

**EQUILIBRIUM:** \( Z_B = \text{tr} \{ e^{-\beta \hat{H}} \} \)
- Quantum disordered system + bath
  - (Quantum) phase transitions
  - Langevin dynamics

  e.g. Bose glass [GIAMARCHI, SCHULZ], [TALK THIBAUD M.]

**OPEN SYSTEMS**
- Quantum disordered system + driving + bath
  - Stationary states
  - Interplay bath-driving

  e.g. [MAIMBOURG, ROSSO et al.]
  [TALK SAPTARSHI M.]

**ISOLATED SYSTEMS**
- Quantum disordered system with unitary dynamics
  - Integrable systems
  - Many-Body Localization (MBL)

  [THIS TALK]
MBL: A tentative definition

“MBL systems are thermodynamically-large, INTERACTING systems (spins, cold atoms, qubits, electrons...) in QUENCHED DISORDER, that evolve with QUANTUM UNITARY DYNAMICS and never reach thermal equilibrium because transport is SUPPRESSED.”

Seminal works:
[ANDERSON 1958]
[ALTSHULER, GEFEN, KAMENEV, LEVITOV 1997]
[BASKO, ALEINER, ALTSHULER 2005]

Reviews:
[ABANIN, ALTMAN, BLOCH, SERBYN 2019 - REV. MOD. PHYS. 91]
[ANNALEN DER PHYSIK: VOL 529, NO 7]
**The Ingredients**

**Unitary Evolution:**

- at $t=0$, pure state: $|\psi(0)\rangle$
- at $t>0$, $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, $\hat{U}(t) = e^{-it\hat{A}}$
- Density matrix: $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$
- Observable $\hat{O}$: $\langle\hat{O}\rangle(t) = \text{tr} \hat{\rho} \hat{O} \hat{\rho}(t)$

Preserves:
- Energy $E = \langle\psi(0)|\hat{A}|\psi(0)\rangle$ - time independence of $\hat{A}$
- "Purity" of state: $\text{tr} [\hat{\rho}^2(t)] = 1$ - Unitarity
- Memory of initial condition - reversible dynamics

No Bath, No Langevin, No Driving, No Dissipation!
**Quenched Randomness:**

\[
\hat{H}_{\text{And}} = \sum_a \varepsilon_a \hat{n}_a - J \sum_{\langle a, b \rangle} (\hat{c}^+_a \hat{c}_b + \hat{c}^+_b \hat{c}_a)
\]

\(\hat{c}^+_a\) creates a fermion at site \(a\).

\(\hat{n}_a = \hat{c}^+_a \hat{c}_a\) counts fermions at site \(a\).

**Local Interactions:**

\[
\hat{H}_{\text{MBL}} = \hat{H}_{\text{And}} + U \sum_{\langle a, b \rangle} \hat{n}_a \hat{n}_b
\]

Interactions are short-ranged in space.
**Quenched Randomness:**

\[ \hat{H}_{\text{And}} = \sum_a \mathbf{E}_a \hat{n}_a - J \sum_{(a,b)} \left( \hat{c}_a^+ \hat{c}_b + \hat{c}_b \hat{c}_a \right) \]

- \( \hat{c}_a^+ \) creates a fermion at site \( a \).
- \( \hat{n}_a = \hat{c}_a^+ \hat{c}_a \) counts fermions at site \( a \).

**Local Interactions:**

\[ \hat{H}_{\text{MBL}} = \hat{H}_{\text{And}} + U \sum_{(a,b)} \hat{n}_a \hat{n}_b \]

**Kinetic Energy**

Random fields \( \mathbf{E}_a \) iid with given \( p(\mathbf{E}) \)

**Notice:** Fermionic model in 1d is equivalent to spin-chain:

\[ \hat{H}_{\text{xxz}} = \sum_a \mathbf{E}_a \hat{\sigma}_a^z - J \sum_{(a,b)} \left( \hat{\sigma}_a^x \hat{\sigma}_b^x + \hat{\sigma}_a^y \hat{\sigma}_b^y \right) + U \sum_{(a,b)} \hat{\sigma}_a^+ \hat{\sigma}_b^- \]

With \( \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \hat{\sigma}_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \), \( \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

(more handy for numerics)
I. Transport and Decay

- **Decay Rates of Local Excitations**: Create local excitation \( \hat{c}^a \) (at site) on top of many-body state \( |\psi_{\text{mb}}\rangle \) of (interacting) particles: \( |\psi(0)\rangle \approx \hat{c}^a |\psi_{\text{mb}}\rangle \). Does it decay (spread out in system), or local memory?

- **Transport of Conserved Quantities** (e.g. energy): Even-out imbalance with diffusive modes?

[Anderson: "Absence of diffusion in certain random lattices"]
I. Transport and Decay

- DECAY RATES OF LOCAL EXCITATIONS: Create local excitation $\hat{a}$ (a-site) on top of many-body state $|\Psi_{MB}\rangle$ of (interacting) particles: $|\psi(0)\rangle = \hat{a}^\dagger |\Psi_{MB}\rangle$
  Does it decay (spread out in system), or local memory?

- TRANSPORT OF CONSERVED QUANTITIES (e.g. energy): Even-out imbalance with diffusive modes?

  [ANDERSON: "Absence of diffusion in certain random lattices"]

II. Growth of quantum correlations

- ENTANGLEMENT ENTROPY: $S_w(t) = -\text{tr}(\hat{\rho}(t) \log \hat{\rho}(t))$ vanishes for pure state $\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)|$
  (stays zero under unitary dynamics)
Subsystems entanglement entropy:

At $t=0$: $|\psi(0)\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

Reduced density matrix:
$$\hat{\rho}_A(t) = \text{Tr}_B \{ \hat{\rho}(t) \} = \text{Tr}_B \{ |\psi(+)\rangle \langle \psi(+) | \}$$

$$S_{AB}(t) = -\text{Tr}_A \{ \hat{\rho}_A(t) \log \hat{\rho}_A(t) \}$$

If system uncoupled, $S_{AB}(t) = 0 \ \forall \ t$. 
Subsystems entanglement entropy:

\[ \rho_A(t) = \text{Tr}_B \{ \hat{\rho}(t) \} = \text{Tr}_B \{ |\psi(t)\rangle \langle \psi(t)| \} \]

\[ S_{AB}(t) = -\text{Tr}_B \{ \log \rho_A(t) \} \]

If system uncoupled, \( S_{AB}(t) = 0 \) \( \forall t \).

III. Emergent Thermal Equilibrium

"Naive" notion of thermalization: \( \hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)| \) \( \xrightarrow{t \to \infty, |\psi(t)\rangle \langle \psi(t)| \to \text{pure state}} \)

\[ \hat{\rho}_B^{\text{pure state}} \xrightarrow{t \to \infty, |\psi(t)\rangle \langle \psi(t)| \to \text{pure state}} \]

"mixture": \( \text{tr} [\hat{\rho}_B^{\text{mixture}}] < 1 \)

Impossible! dynamics preserves purity (unitarity) and memory of \( |\psi(0)\rangle \) (reversibility)
Better notion: thermalization of local subsystems

\[ \hat{O}_A = \text{local observable (operator) in } A. \]
\[ \text{e.g. } \hat{\eta}_a \text{ for site } a \in A \]

Thermal expectation value:
\[ E_\beta [\hat{O}_A] = \frac{1}{Z_\beta} \text{ Tr} \left\{ e^{-\beta \hat{H}} \hat{O}_A \right\} \]

Thermalization if:
\[ \lim_{L \to \infty} \lim_{t \to \pm \infty} \text{ Tr} \left\{ \hat{O}_A \hat{\rho}(t) \right\} = E_\beta [\hat{O}_A] \]
with \( \beta \) fixed by \( \langle \psi(t)|\hat{H}|\psi(t)\rangle = E = E_\beta [\hat{H}] \)

"The system is a bath for itself, i.e. for any local subsystem."
Better notion: thermalization of local subsystems

\[ \hat{O}_A = \text{local observable (operator) in } A. \]

e.g. \( \hat{n}_a \) for site \( a \in A \)

Thermal expectation value:

\[ E_{\beta} [\hat{O}_A] = \frac{1}{Z_{\beta}} \langle \psi(0) | e^{-\beta \hat{H}} \hat{O}_A | \psi(0) \rangle \]

Thermalization if:

\[ \lim_{L \to \infty} \lim_{t \to \infty} \text{tr} \left[ \hat{O}_A \hat{\rho}(t) \right] = E_{\beta} [\hat{O}_A] \]

(e fixed)

with \( \beta \) fixed by \( \langle \psi(0) | \hat{H} | \psi(0) \rangle = E = E_{\beta} [\hat{H}] \)

"The system is a bath for itself, i.e. for any local subsystem."

Notice: most of these questions can be formulated as properties of the eigenstates/eigenspectrum:

\[ \hat{A}_\alpha \ket{\psi_k} = E_{\alpha} \ket{\psi_k} \]

(e.g. ETH = eigenstate thermalization hypothesis)
The Picture  

from numerics on small system-sizes, approximate  
analytics, very few rigorous results...

\[ \hat{H}_{MB} = W \sum_a E_a \hat{N}_a - J \sum_{\langle a,b \rangle} (\hat{C}_a \hat{C}_b + \hat{C}_b \hat{C}_a) + U \sum_{\langle a,b \rangle} \hat{N}_a \hat{N}_b \]

MBL  
DELOCALIZED

Local excitations do not decay (local memory)  
Vanishing transport coefficients \( \sigma_x, D = 0 \)  
Slow growth of quantum correlations \( S_{xx}(t) \sim \log t \)  
Local observables do not converge  
to thermal expectation value

v.s.

Excitations have decay rate \( \Gamma > 0 \)  
v.s.

Diffusion (or superdiffusion)  
Fast entanglement growth \( S_{xx}(t) \sim t \)  
The system looks locally like a thermal mixture.

CAN NOT USE STATISTICAL ENSEMBLES  
TO CHARACTERIZE SYSTEM AT \( t \gg 1 \).

DYNAMICAL TRANSITION: NO THEORY!
MBL is not a k70 Glass

- Phase is dynamical, not thermodynamical - not related to structure of Boltzmann measure, RSB etc.

  [In the language of phase transition, MBL is the unbroken-symmetry phase, symmetry is TIME REVERSAL. Order parameter: distribution of a function.]
**MBL is not a k70 Glass**

- Phase is dynamical, not thermodynamical - not related to structure of Boltzmann measure, RSB etc.
  
  [In the language of phase transition, MBL is the unbroken-symmetry phase, symmetry is **TIME REVERSAL**. Order parameter: distribution of a function.]

- MBL is not a metastable state: not due to (free-) energy barriers slowing down equilibration. Rather, clue to presence of local conservation laws. *[IMBRIE, ROS, SCARDICCHIO 2017]*
**MBL is not a k>0 Glass**

- Phase is dynamical, not thermodynamical - not related to structure of Boltzmann measure, RSB etc.
  
  [In the language of phase transition, MBL is the unbroken-symmetry phase, symmetry is \textit{TIME REVERSAL}. Order parameter: distribution of a function.]

- MBL is not a metastable state: not due to (free-) energy barriers slowing-down equilibration. Rather, clue to presence of local conservation laws. [IMBRIE, ROS, SCARDICCHIO 2017]

- Not restricted to low energy of many-body state (i.e., ground state, T=0): quantumness matters at $E \gg E_s$ here.

- In a glass, diffusion is there. Transport properties are different. Quantum localization.
I. MBL is not a quantum glass.

- Quantum Disordered Systems: Frameworks
- MBL: A tentative definition
  - The Ingredients
  - The Questions
  - The Picture
- MBL is not a glass with $h > 0$.

II. Localization is glassy, though.

- Computing Decay Rates:
- The directed polymer problem
- Localization is glassy, though
Computing Decay Rates: the simplest example

- Anderson problem: a single particle on a lattice, at a site "a" at $t=0$.

$$H_{\text{And}} = \sum_{\epsilon_a} \epsilon_a \hat{a}_a - J \sum_{\langle a,b \rangle} c_a^\dagger c_b + c_b^\dagger c_a \iff \hat{H}^{(a)} = \sum_{\epsilon_a} \epsilon_a \hat{a}_a - J \sum_{\langle a,b \rangle} \langle a | b \rangle + \langle b | a \rangle$$

$|a\rangle$ = wave function localized at site "a": $|\psi(0)\rangle = |a\rangle$

- Localization is encoded in properties of the LOCAL SELF ENERGIES $S_a(\epsilon)$:

$$\langle a | \frac{1}{\epsilon - H^{(a)}} | a \rangle = \frac{1}{\epsilon - W\epsilon_a - S_a(\epsilon)}$$

$\uparrow$shift due to presence of $J$

The decay rate of particle at "a" at $t=0$ is $\propto \lim_{\eta \to 0} \text{Im} \{S_a(E + i\eta)\}$

In the localized phase, $\lim_{\eta \to 0} \text{Im} \{S_a(E + i\eta)\} = 0$ in distribution.
Distribution of local self-energies is hard to compute. On Bethe lattice, cavity equations for cavity self-energies $S_a^\text{cav}(z)$:

$$S_a(z) = \frac{1}{z - \epsilon_a - \sum_{\text{ne}} S_a^\text{cav}(z)} \quad (*)$$

[Abou-Chacra, Anderson, Thouless 1973]

Self-consistent description of localized phase:

(i) Assume localization of isolated system: $\text{Im} \{S_a^\text{cav}(E)\} = 0$

(ii) Open system at boundary: infinitesimal decay rate $\eta \ll 1$

(iii) Assume localization: $\eta$ induces small decay rate in bulk ($\text{Im} S_a^\text{cav} \ll 1$): linearize $(*)$

$$\text{Im} \{S_a\} = \sum_{\Phi: \alpha \to \partial R} \prod_{\text{sep}} \frac{J^2}{[E - \epsilon_s - \text{Re} S_a^\text{cav}(E)]^2} \cdot \eta$$

response $\Gamma^2$
Localization: \[ \Gamma^2(E) = \sum_{\Phi: a \to B_R} \prod_{\text{sep}} \frac{J^2}{[E - W - \text{Re} \; S^w_0(E)]^2} \sim e^{-R \gamma(w;J,E) + O(R)} \]

Localization stable when \( \gamma(w;J,E) > 0 \).

The directed polymer problem

Directed polymer on Bethe lattice: \[ Z^\beta = \sum_{\text{paths} \Phi} \prod_{|\Phi| = R} e^{-\beta E_\Phi} \sim e^{-R^\beta} \]

This problem has a glass transition at \( \beta_c \):

\( \beta < \beta_c \): \( Z ^\beta \) contributed by \( N = e^{R^\beta + O(R)} \) paths. Positive entropy \( S > 0 \).

\( \beta > \beta_c \): Freezing: \( S = 0 \). Boltzmann measure condenses on sub-exponential number of paths. Replica Symmetry Breaking.
Localization is glassy, though

- When the system is localized, underlying polymer is in which phase?
  
  For single particles: localization is always in frozen phase (glassy).
  
  System is glassy up to the transition point to delocalization.

- In interacting systems with heterogeneous (slow/fast) d.o.f.,
  can have localized regimes that are not frozen.

  Interplay between localization & glassyness is rich!
Thanks!

Reviews:
[ABANIN, ALTMAN, BLOCH, SERBYN 2019 - REV. MOD. PHYS. 91]
[ANNALEN DER. PHYSIK: VOL 529, NO 7]