What is the appropriate theory?

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How to find the appropriate theory?

- no clue what the theory is?
- try to measure it!
- here: disordered elastic systems

Contact line wetting

(C) E. Rolley

- isobutanol on a randomly silanized
height jumps = avalanches

$\bar{h}_{Lc} (\mu m)$ spatially averaged height

 avalanches

$w (\mu m)$ plate position
Theory

Equation of motion (for SR elasticity for simplicity)

\[ \eta \partial_t u(x, t) = c \nabla^2 u(x, t) + m^2 [w - u(x, t)] + F(x, u(x, t)) \]

Forces are drawn from a Gaussian, and have correlations

\[ \overline{F(x, u)F(x', u')}^c = \delta^d(x - x') \Delta(u - u') \]

Field theory (MSR=classical limit \( \hbar \to 0 \) of Keldysh)

\[ \mathcal{S}[\tilde{u}, u] = \int_{x, t} \tilde{u}(x, t) \left[ \eta \partial_t u(x, t) - c \nabla^2 u(x, t) + m^2 (u(x, t) - w) \right] \]

\[ -\frac{1}{2} \int_{x, t, t'} \tilde{u}(x, t) \tilde{u}(x, t') \Delta (u(x, t) - u(x, t')) \]

height of the interface \( w = vt \)

will be measured
### Why did we measure $\Delta$?

**Action**

$$\mathcal{S}[^\tilde{u}, u] = \int_{x,t} \tilde{u}(x, t) \left[ \eta \partial_t u(x, t) - c \nabla^2 u(x, t) + m^2 \left( u(x, t) - w \right) \right]$$

$$-\frac{1}{2} \int_{x,t,t'} \tilde{u}(x, t)\tilde{u}(x, t') \Delta \left( u(x, t) - u(x, t') \right)$$

**Center of mass at large $t$, i.e. $\omega \to 0$**

$$u_w := \lim_{t \to \infty} \frac{1}{L^d} \int_x u(x, t) \bigg|_{w}$$

$$\Delta(w - w') \equiv \Gamma^{(2)} = \mathcal{L} \circ u_w^\omega u_{w'}^c = \left[ \mathcal{R}^{-1} \right]^2 u_w^c u_{w'}^c = (m^2)^2 u_w^c u_{w'}^c$$

**Legendre transform**

**Amputate 2-point function (response)**
Renormalization in DNA-unzipping

**Graphs and Diagrams**

1. **Graph**:
   - Title: Unzipping - Rezipping
   - X-axis: $w$ [nm]
   - Y-axis: $F_w$ [pN]
   - Legend: Unzipping (red), Rezipping (blue)
   - The graph shows two curves for unzipping and rezipping, indicating a transition between states.

2. **Diagram**:
   - Optical Trap
   - ssDNA
   - dsDNA
   - Handle
   - Micropipette
   - Axes: $w$ and $u$
   - The diagram illustrates the setup for experimental measurements.

3. **Equation**:
   - $m^{-2} [\text{nm/pN}]:$ inverse of confining trap strength $m^2$

**Text Excerpts**

- "...critical depinning (non-case described by the Sinai model."
- "The FRG arises as the field theory approach to disordered systems for interfaces (FRG)."
- "The slope of each segment, equivalent to the effective stiffness, decreases during the experiment, permitting us to measure the scaling of ..."
- "The shape function. The scaling relation for a stiffness of the optical trap of about ... typically in the upper critical dimension, here parameterised by $m$."
- "This result agrees with the experimentally measured values, which range from a few tens to hundreds of basepairs released and absorbed in the rips has been previously measured."
- "The hairpin stiffness of one nucleotide. Modeling the elastic response of the molecular construct (ssDNA, dsDNA) attached, while ..."
scaling collapse with $\zeta = \frac{4}{3}$
Imbibition

The Tang-Leschhorn cellular automaton of 1992 TL92

variants: Buldyrev, S. Havlin and H.E. Stanley 1992
Anharmonic depinning respects the Middleton theorem

$$\mathcal{E}_{el} = \sum_{i} \frac{c_4}{4} (u_i - u_{i+1})^4$$

anharmonic depinning respects the Middleton theorem

= return point memory (not guaranteed for qKPZ)
TL92 and directed percolation ($d = 1$)

\[ f_c := m^2(u - w) \]

Used for driving
2-point function

\[
\frac{1}{2} \left[ u(x) - u(y) \right]^2 \sim \begin{cases} 
A |x - y|^{2\zeta}, & |x - y| < \xi \\
B m^{-2\zeta_m}, & |x - y| > \xi 
\end{cases}
\]

from directed percolation

\[
\zeta_{d=1} = \frac{\nu_\perp}{\nu_\parallel} = 0.632613(3)
\]

\[
\zeta_{m} = \frac{2\nu_\perp}{1 + \nu_\perp} = 1.046190(4)
\]

two distinct exponents in all \(d\)

\[
\zeta_m > \zeta
\]
What is the appropriate long-distance theory? Can we measure it?

\[ \eta \partial_t u(x, t) = c \nabla^2 u(x, t) + c_4 \nabla [\nabla u(x, t)]^3 - m^2 [u(x, t) - w] + F(x, u(x, t)) \]

- standard elasticity
- non-linear elasticity
- disorder force
- confining potential
- background field
What is the appropriate long-distance theory?
Can we measure it?

\[ \eta \partial_t u(x, t) = c \nabla^2 u(x, t) + c_4 \nabla \left[ \nabla u(x, t) \right]^3 - m^2 [u(x, t) - w(x, t)] \\
+ \lambda \left[ \nabla u(x, t) \right]^2 + F(x, u(x, t)) \]

- standard elasticity
- non-linear elasticity
- KPZ term
- disorder force
- confining potential (unrenormalized)
- background field (modulated)
Measuring the elastic constants for harmonic depinning (qEW)

\[ \eta \partial_t u(x, t) = c \nabla^2 u(x, t) - m^2[u(x, t) - w(x, t)] + F(x, u(x, t)) \]

\[ w(x) = w_0 + A \sin\left(\frac{\pi x}{L}\right) \]

qEW

(standard elasticity)
The projections on these modes. To be specific, write, with

We then measure the mean interface profile

After each avalanche, we increase

with a spatially modulated background field

as much information as possible to constrain the field theory.

A similar argument for

sites

FIG. 13. We drive the interface with a spatially modulated driving.

The scaling relation for

\[ \text{in agreement with their expressions in Eqs. (65)} \]

Measuring the elastic

anharmonic depinning \((c_4 > 0)\)

\[ \ln c \]

\[ \ln \lambda \]

\[ \ln(m) \]

\[ m^{-\Psi_c} \]

\[ m^{-\Psi_\lambda} \]
Measuring the effective force correlator

\[ \Delta(w - w') = m^4 L^d (u_w - w) (u_{w'} - w')^c \]

\[ u_w = \frac{1}{L^d} \int_x u_w(x) \]

centre-of-mass position given \( w \)

\[ \Delta(w) \]

\( d = 1 \)

FIG. 12. (Left) Correlators in \( d = 1 \) from simulations of harmonic depinning (qEW) and anharmonic depinning (in the qKPZ universality class), compared to the analytic solution of the flow equations. We see that the qKPZ FRG one loop solution is around 3 times closer to the numerical simulation than qEW one-loop to his, highlighting the efficiency of our procedure.

FIG. 13. (Left) Correlators in \( d = 2 \) from simulations of harmonic depinning (qEW) and anharmonic depinning (qKPZ class), compared to the solution of the FRG flow equations. The FRG solution is much closer to anharmonic depinning than to qEW. The correlators are rescaled such that \( \Delta(0) = |\rho(0 + 0)| = 1 \). (Right) Difference of the rescaled correlators measured and analytical. The agreement between simulations and theory is of the same order of magnitude for the two universality class, even if the qKPZ theory is much more sophisticated.

\[ \Delta(w) \]

\( d = 2 \), and \( d = 3 \), we constructed a consistent theory. The crucial ingredient is a flow-equation for the KPZ non-linearity, which is controlled in dimension \( d \). Behind this feature lies the observation that all field theories for qEW with SR or LR elasticity, as well as qKPZ merge into a single theory in dimension \( d = 0 \). Our theory has predictive powers as long as we have a sufficient knowledge of the qEW fixed point in small dimensions, and we are not too far away from \( d = 0 \).

We derived several bounds, respected in low dimensions, but violated in dimension \( d = 3 \); there we currently can only close our scheme with an adhoc assumption.

We hope that our method of first measuring the effective theory in a simulation, before attempting to build a field theory, can serve in other contexts as well, as e.g. fully developed turbulence. Applying our approach to other growth experiments in \( d = 2 \) for which no theory is available seems promising. We hope it will also shed light on the problems in the standard (thermal) KPZ equation in higher dimensions.
FIG. 14. Computation of the effective $c$ and $\lambda$ for the qEW equation. Apart from large $m$, the effective elasticity $c$ is unrenormalized, as predicted by Statistical Tilt Symmetry. The measured non-linearity vanishes.

FIG. 15. Left: Scaling of $c$ and $\lambda$ for anharmonic depinning in 1d. Right: Measured amplitude ratios $A$ for TL92 and anharmonic depinning. The second-order polynomial fits show convergence to $A \approx 1$. We checked that higher-order relations (given in Appendix B) give the same results. We further checked that the results given for $m \to 0$ are the same as those obtained as a response to a tilt. (Note that to introduce a tilt with our driving protocol, one has to tilt both the driving potential and the interface).

The determination of the effective parameters $c$ and $\lambda$ is not the only application of this algorithm: one could measure the effective decay of subleading parameters present at the beginning of the flow, such as $c^4_2$, and obtain valuable information on the crossover to the qKPZ universality class. This could be helpful for experiments and is summarized in Appendix B. This technique is mostly limited by computer resources.

G. The universal amplitude ratio $A$ An important question is whether qKPZ is the proper large-distance description of TL92, anharmonic depinning, and itself. To ensure this the properly renormalized non-linearity needs to flow to a fixed point. This is achieved by defining the universal amplitude ratio $A$ as

$$A := \frac{\lambda}{\Delta(0)} \equiv \frac{|\Delta'(0^+)|}{|c|}$$

On the other hand

$$\mathcal{A} := \rho \frac{\ln(m)}{c} \equiv \frac{\Delta(0)}{|\Delta'(0^+)|} \frac{\lambda}{c}$$

For $m \to 0$.

VI. CONCLUSION

We showed through theoretical arguments and numerical checks, that anharmonic depinning, qKPZ and the cellular automaton TL92 are in the same universality class, the qKPZ universality class. We then elucidated the scaling relations for driving through a parabolic confining potential. This allowed us to understand statics and dynamics of our system. Finally we developed an algorithm to measure the renormalized (effective) coefficients of the continuity equation. This will be useful to constrain, and finally construct the field theory presented in a sequel of this work.

We believe that our technique to extract the effective coupling constants by measuring the static response of the system under spatially modulated perturbations may yield important information in other systems that lack a proper field theoretic description. We started to extend it to the KPZ equation itself.

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We thank Alberto Rosso for useful discussions.

Measuring the effective force correlator

\[ \Delta(w - w') = m^4 L^d (u_w - w)(u_{w'} - w') \]

\[ u_w = \frac{1}{L^d} \int_x u_w(x) \]

Centre-of-mass position given \( w \)

Almost indistinguishable in \( d = 1 \)
Universality classes for depinning

qKPZ SR-elasticity
- $d = 4$
- $d = 3$
- $d = 2$
- magnetic domain wall
- $d = 1$
- imbibition

qEW SR-elasticity
- $d = 4$
- $d = 3$
- $d = 2$
- vortex lattice/CDW
- $d = 1$
- magnetic domain wall

qEW LR-elasticity
- $d = 2$
- magnetic domain walls, earthquakes, knitting
- $d = 1$
- contact line, fracture
- $d = 0$
- analytically solvable: dragged particle (RNA/DNA peeling)
FRG flow equations

Flow of the disorder for qKPZ

\[ \partial_\varepsilon \tilde{\Delta}(u) = \left( 4 - d \frac{\zeta_m}{\zeta} - 2\zeta_m \right) \tilde{\Delta}(u) + u\zeta_m \tilde{\Delta}'(u) \]

\[ + \frac{d(d + 2)}{12} \tilde{\lambda}^2 \tilde{\Delta}(u)^2 - \tilde{\Delta}'(u)^2 - \tilde{\Delta}''(u) \left[ \tilde{\Delta}(u) - \tilde{\Delta}(0) \right] \]

replace \( \zeta_m / \zeta \)

\[ \frac{\zeta_m}{\zeta} = 1 + \frac{1}{2} \left[ -\tilde{\lambda} \tilde{\Delta}'(0^+) - \frac{d - 1}{3} \tilde{\lambda}^2 \tilde{\Delta}(0) \right]. \]

flow for \( \lambda \) (with confining potential, i.e. massive theory)

\[ -m d_\lambda \tilde{\lambda} = \zeta_m \tilde{\lambda} - \frac{4 - d}{6} \tilde{\lambda}^3 \tilde{\Delta}(0) \quad \Rightarrow \quad \tilde{\lambda}_c = \sqrt{\frac{6\zeta_m}{(4 - d)\tilde{\Delta}(0)}} \]
The anomalous dimension reads

\[ \zeta_m = \frac{c}{\epsilon^{1/2}} \]

\[ \zeta_m = 0.86 \]

\[ \zeta = 0.69 \]

\[ \zeta^d = 1.27 \]

\[ A^d = 1.27 \]

\[ \zeta_m = 1.05 \]

\[ \zeta = 0.63 \]

\[ \zeta^d = 1.10(2) \]

\[ A^d = 1.10(2) \]
FIG. 12. (Left) Correlators in $d=1$ from simulations of harmonic depinning (qEW) and anharmonic depinning (in the qKPZ universality class), compared to the analytic solution of the flow equations. The correlators are rescaled such that $(0) = |0(0 + \Delta)| = 1$. (Right) Difference of the rescaled correlators measured or analytical. We see that the qKPZ FRG one loop solution is around 3 times closer to the numerical simulation than qEW one-loop to his, highlighting the efficiency of our procedure.

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Conclusions

• when in doubt: measure effective long-distance action (\(= \text{theory/description}\))
• standard elastic depinning (\(\text{qEW}\)) has non-trivial disorder correlator given by FRG
• imbibition (e.g. TL92), anharmonic depinning and \(\text{qKPZ}\) all belong to the same universality class: the effective long-wavelength theory is \(\text{qKPZ}\)
• you need to introduce a confining potential \(m^2[w - u(x, t)]\) to measure disorder correlations
  \(\Rightarrow\) give up the Cole-Hopf transform
  \(\Rightarrow\) yields an RG fixed point
• a field theory can be build
Abstract. Domain walls in magnets, vortex lattices in superconductors, contact lines at depinning, and many other systems can be modeled as an elastic system subject to quenched disorder. The ensuing field theory possesses a well-controlled perturbative expansion around its upper critical dimension. Contrary to standard field theory, the renormalization group flow involves a function, the disorder correlator \( \Delta(w) \), and is therefore termed the functional renormalization group (FRG). \( \Delta(w) \) is a physical observable, the auto-correlation function of the center of mass of the elastic manifold. In this review, we give a pedagogical introduction into its phenomenology and techniques. This allows us to treat both equilibrium (statics), and depinning (dynamics). Building on these techniques, avalanche observables are accessible: distributions of size, duration, and velocity, as well as the spatial and temporal shape. Various equivalences between disordered elastic manifolds, and sandpile models exist: an elastic string driven at a point and the Oslo model; disordered elastic manifolds and Manna sandpiles; charge density waves and Abelian sandpiles or loop-erased random walks. Each of the mappings between these systems requires specific techniques, which we develop, including modeling of discrete stochastic systems via coarse-grained stochastic equations of motion, super-symmetry techniques, and cellular automata. Stronger than quadratic nearest-neighbor interactions lead to directed percolation, and non-linear surface growth with additional KPZ terms. On the other hand, KPZ without disorder can be mapped back to disordered elastic manifolds, either on the directed polymer for its steady state, or a single particle for its decay. Other topics covered are the relation between functional RG and replica symmetry breaking, and random field magnets. Emphasis is given to numerical and experimental tests of the theory.